

Multipermutation Ulam Sphere Analysis Toward Characterizing Maximal Code Size

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Abstract—Permutation codes, in the form of *rank modulation*, have shown promise for applications such as flash memory. One of the metrics recently suggested as appropriate for rank modulation is the Ulam metric, which measures the minimum translocation distance between permutations. Multipermutation codes have also been proposed as a generalization of permutation codes that would improve code size (and consequently the code rate). In this paper we analyze the Ulam metric in the context of multipermutations, noting some similarities and differences between the Ulam metric in the context of permutations. We also consider sphere sizes for multipermutations under the Ulam metric and resulting bounds on code size.

I. INTRODUCTION

Permutation (and multipermutation) codes were invented as early as the 1960's, when Slepian proposed constructing a code by permuting the order of the numbers of an initial sequence [13]. More recently, Jiang et al. proposed permutation codes utilizing the Kendall- τ metric for use in flash memory via the *rank modulation* scheme [8]. Since then, permutation codes and their generalization to multipermutation codes have been a hot topic in the research community with various related schemes being suggested [1], [2], [3], [4], [9], [11].

One scheme of particular interest was the proposal of Farnoud et al. to utilize the Ulam metric in place of the Kendall- τ metric [3] and subsequent study expounded upon code size bounds [7]. The Ulam metric measures the minimum number of translocations needed to transform one permutation into another, whereas the Kendall- τ metric measures the minimum adjacent transpositions needed to transform one permutation into another. Errors in flash memory devices occur when cell charges leak or when rewriting operations cause overshoot errors resulting in inaccurate charge levels. While the Kendall- τ metric is suitable for correcting relatively small errors of this nature, the Ulam metric would be more robust to large charge leakages or overshoot errors within a cell.

However, there is a trade-off in code size when rank modulation is used in conjunction with the Ulam metric instead of the Kendall- τ metric. The Ulam distance between permutations is always less than or equal to the Kendall- τ distance between permutations, which implies that the maximum code size for a permutation code utilizing the Ulam metric is less than or equal to the maximum code size of a permutation code utilizing the Kendall- τ metric [3]. One possible compensation for this trade-off is the generalization from permutation codes

to multipermutation codes, which improves the maximum possible code size [4].

In flash memory devices, permutations or multipermutations may be modeled physically by relative rankings of cell charges. The number of possible messages is limited by the number of distinguishable relative rankings. However, it was shown in [4] that multipermutations may significantly increase the total possible messages compared to ordinary permutations. For example, if only k different charge levels are possible, then permutations of length k can be stored. Hence, in r blocks of length k , one may store $(k!)^r$ potential messages. On the other hand, if one uses r -regular multipermutations in the same set of blocks, then $(kr)!/(r!)^k$ potential messages are possible.

Bounds on permutation codes in the Ulam metric were studied in [3] and [7]. In [10], the nonexistence of nontrivial perfect permutation codes in the Ulam metric was proven by examining the size of Ulam spheres, spheres comprised of all permutations within a given Ulam distance of a particular permutation. However, no similar study of Multipermutation Ulam spheres exists, and currently known bounds on code size do not always consider the problem of differing sphere sizes. The current paper examines Ulam sphere sizes in the context of multipermutations and provides new bounds on code size.

The paper is organized as follows: First, Section II defines notation and basic concepts used in the paper. Next, Section III compares properties of the Ulam metric as defined for permutations and multipermutations, and then provides a simplification of the r -regular Ulam metric for multipermutations (Lemmas 1 and 2). Section IV considers an application of Young Tableaux and the RSK-correspondence to calculate r -regular Ulam sphere sizes (Lemma 5 and Prop. 6). Section V then discusses duplicate translocation sets and a method of calculating the size of spheres of radius $t = 1$ for any center (Thm. 13). Section VI follows, demonstrating minimal and maximal sphere sizes (Lemmas 14 and 16) and providing both lower and upper bounds on code size (Lemmas 15, 17, and 18). Finally Section VII gives some concluding remarks.

II. PRELIMINARIES AND NOTATION

In this section we introduce notation and definitions used in this paper. Unless otherwise stated, definitions are based on conventions established in [3], [4], and [10]. Throughout this paper n and r are assumed to be positive integers, r dividing n .

The notation $[n]$ denotes the set $\{1, 2, \dots, n\}$ and \mathbb{S}_n denotes the set of permutations on $[n]$, i.e. the symmetric group of size $n!$. For $\sigma \in \mathbb{S}_n$, we write $\sigma = [\sigma(1), \sigma(2), \dots, \sigma(n)]$, where for all $i \in [n]$, $\sigma(i)$ is the image of i under σ . Throughout this paper we assume $\sigma, \pi \in \mathbb{S}_n$. With a slight abuse of notation, we may also use σ to mean the sequence $(\sigma(1), \sigma(2), \dots, \sigma(n)) \in \mathbb{Z}^n$ associated with $\sigma \in \mathbb{S}_n$. Multiplication of permutations is defined by composition so that for all $i \in [n]$, we have $(\sigma\tau)(i) = \sigma(\tau(i))$. The identity permutation, $[1, 2, \dots, n] \in \mathbb{S}_n$ is denoted by e .

An r -regular multiset is a multiset such that each of its elements is repeated r times. A *multipermutation* is an ordered tuple of the elements of a multiset, and in the instance of an r -regular multiset, is called an r -regular *multipermutation*. Following the work of [4], this study focuses on r -regular multipermutations, although many results are extendible to general multipermutations.

For each $\sigma \in \mathbb{S}_n$ we define a corresponding r -regular multipermutation \mathbf{m}_σ^r as follows: for all $i \in [n]$ and $j \in [n/r]$,

$$\mathbf{m}_\sigma^r(i) := j \text{ if and only if } (j-1)r + 1 \leq \sigma(i) \leq jr,$$

and $\mathbf{m}_\sigma^r := (\mathbf{m}_\sigma^r(1), \mathbf{m}_\sigma^r(2), \dots, \mathbf{m}_\sigma^r(n)) \in \mathbb{Z}^n$. For example, if $n = 6$, $r = 2$, and $\sigma = [1, 5, 2, 4, 3, 6]$, then $\mathbf{m}_\sigma^r = (1, 3, 1, 2, 2, 3)$. This definition differs slightly from the correspondence defined in [4], which was defined in terms of the inverse permutation. This is so that properties of the Ulam metric for permutations will carry over to the Ulam metric for multipermutations (Lemmas 1 and 2 of Section III).

With the correspondence above, we may define an equivalence relation between elements of \mathbb{S}_n . We say that $\sigma \equiv_r \pi$ if and only if $\mathbf{m}_\sigma^r = \mathbf{m}_\pi^r$. The equivalence class $R_r(\sigma)$ of $\sigma \in \mathbb{S}_n$ is defined as $R_r(\sigma) := \{\pi \in \mathbb{S}_n : \pi \equiv_r \sigma\}$. For a subset $S \subseteq \mathbb{S}_n$, the notation $\mathcal{M}_r(S) := \{\mathbf{m}_\sigma^r : \sigma \in S\}$, i.e. the set of r -regular multipermutations corresponding to elements of S .

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The following definition is our own. For any $\mathbf{m} \in \mathbb{Z}^n$, and $\sigma \in \mathbb{S}_n$, we define the product (a right group action) $\mathbf{m} \cdot \sigma$ by composition, similarly to the definition of multiplication of permutations. More precisely, for all $i \in [n]$, let $(\mathbf{m} \cdot \sigma)(i) := \mathbf{m}(\sigma(i))$. It is easily confirmed that $\mathbf{m} \cdot e = \mathbf{m}$ for all $\mathbf{m} \in \mathbb{Z}^n$. It is also easily confirmed that for all $\sigma, \pi \in \mathbb{S}_n$, we have $\mathbf{m} \cdot (\sigma\pi) = (\mathbf{m} \cdot \sigma) \cdot \pi$. With this definition, notice that $\mathbf{m}_\sigma^r \cdot \pi = \mathbf{m}_{\sigma\pi}^r$. It is possible for different permutations to correspond to the same multipermutation, but for $\tau \in \mathbb{S}_n$, it is clear that $\mathbf{m}_\sigma^r = \mathbf{m}_\pi^r$ implies $\mathbf{m}_\sigma^r \cdot \tau = \mathbf{m}_\pi^r \cdot \tau$.

We finish this section by defining what a multipermutation code is. A subset $C \subseteq \mathbb{S}_n$ is called an r -regular multipermutation code if and only if for all $\sigma \in C$, we also have $R_r(\sigma) \subseteq C$. Such a code is denoted by $\text{MPC}(n, r)$, and we say that C is an $\text{MPC}(n, r)$. If C is an $\text{MPC}(n, r)$ then whenever a permutation is a member of C its entire

equivalence class is also contained within C . Thus if C is an $\text{MPC}(n, r)$ it can be represented by the set of r -regular multipermutations associated with elements of C , i.e. the set $\mathcal{M}_r(C)$. Moreover, we define the cardinality $|C|_r$ of an $\text{MPC}(n, r)$ C as $|C|_r := |\mathcal{M}_r(C)|$ (this notation and definition differs slightly from [4]).

III. MULTIPERMUTATION ULAM METRIC

In this section we discuss some similarities and differences between the Ulam metric for permutations and the Ulam metric for multipermutations. We begin by defining the Ulam metric for permutations.

For any two sequences $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$, $\ell(\mathbf{u}, \mathbf{v})$ denotes the length of the longest common subsequence of \mathbf{u} and \mathbf{v} . In other words, $\ell(\mathbf{u}, \mathbf{v})$ is the largest integer $k \in \mathbb{Z}_{>0}$ such that there exists a sequence (a_1, a_2, \dots, a_k) where for all $p \in [k]$, we have $a_p = \mathbf{u}(i_p) = \mathbf{v}(j_p)$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and $1 \leq j_1 < j_2 < \dots < j_k \leq n$. The *Ulam distance* $d_o(\sigma, \pi)$ between permutations $\sigma, \pi \in \mathbb{S}_n$ is defined as $d_o(\sigma, \pi) := n - \ell(\sigma, \pi)$.

It is also known that the Ulam distance $d_o(\sigma, \pi)$ between $\sigma, \pi \in \mathbb{S}_n$ is equivalent to the minimum number of translocations needed to transform σ into π [3]. Here, for distinct $i, j \in [n]$, the translocation $\phi(i, j) \in \mathbb{S}_n$ is defined as follows:

$$\phi(i, j) := \begin{cases} [1, 2, \dots, i-1, i+1, i+2, \dots, j, i, j+1, \dots, n] & \text{if } i < j \\ [1, 2, \dots, j-1, i, j, j+1, \dots, i-1, i+1, \dots, n] & \text{otherwise} \end{cases}$$

The notation $\phi(i, i)$ is understood to mean the identity permutation, e . When it is not necessary to specify any index, a translocation may be written simply as ϕ . Intuitively, when multiplied on the right of a permutation $\sigma \in \mathbb{S}_n$, the translocation $\phi(i, j) \in \mathbb{S}_n$ deletes $\sigma(i)$ from the i th position of σ and then inserts it in the new j th position (shifting positions between i and j in the process).

The r -regular *Ulam distance* $d_o^r(\sigma, \pi)$ between permutations $\sigma, \pi \in \mathbb{S}_n$ is defined as the minimum Ulam distance among all members of $R_r(\sigma)$ and $R_r(\pi)$. That is, $d_o^r(\sigma, \pi) := \min_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} d_o(\sigma', \pi')$. Notice that the r -regular Ulam distance is defined over equivalence classes.

Although technically a distance between equivalence classes, it is convenient to think of the r -regular Ulam distance instead as a distance between multipermutations. Viewed this way, the property of the Ulam metric for permutations, that it can be defined in terms of longest common subsequences or equivalently in terms of translocations, carries over to the r -regular Ulam distance. The next lemma states that the r -regular Ulam distance between permutations σ and π is equal to n minus the length of the longest common subsequence of their corresponding r -regular multipermutations.

Lemma 1. $d_o^r(\sigma, \pi) = n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$.

Proof: We will first show that $d_o^r(\sigma, \pi) \geq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. By definition of $d_o^r(\sigma, \pi)$, there exist $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ such that $d_o^r(\sigma, \pi) = d_o(\sigma', \pi') = n - \ell(\sigma', \pi')$. Hence if for all $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ we have $\ell(\sigma', \pi') \leq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$, then $d_o^r(\sigma, \pi) \geq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$ (subtracting a larger value from n results in a smaller overall value). Therefore it suffices to show that for all $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$, that $\ell(\sigma', \pi') \leq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. This is simple to prove because if two permutations have a common subsequence, then their corresponding r -regular multipermutations will have a related common subsequence. Let $\sigma' \in R_r(\sigma)$, $\pi' \in R_r(\pi)$, and $\ell(\sigma', \pi') = k$. Then there exist indexes $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and $1 \leq j_1 < j_2 < \dots < j_k \leq n$ such that for all $p \in [k]$, $\sigma'(i_p) = \pi'(j_p)$. Of course, whenever $\sigma'(i) = \pi'(j)$, then $\mathbf{m}_{\sigma'}^r(i) = \mathbf{m}_{\pi'}^r(j)$. Therefore $\ell(\sigma', \pi') = k \leq \ell(\mathbf{m}_{\sigma'}^r, \mathbf{m}_{\pi'}^r) = \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$.

Next, we will show that $d_o^r(\sigma, \pi) \leq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. Note that

$$\begin{aligned} d_o^r(\sigma, \pi) &= \min_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} d_o(\sigma', \pi') \\ &= \min_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} (n - \ell(\sigma', \pi')) \\ &= n - \max_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} \ell(\sigma', \pi'). \end{aligned}$$

Here if $\max_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} \ell(\sigma', \pi') \geq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$, then $d_o^r(\sigma, \pi) \leq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$ (subtracting a smaller value from n results in a larger overall value). It is enough to show that there exist $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ such that $\ell(\sigma', \pi') \geq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. To prove this fact, we take a longest common subsequence of \mathbf{m}_σ^r and \mathbf{m}_π^r and then carefully choose $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ to have an equally long common subsequence. The next paragraph describes how this can be done.

Let $\ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) = k$ and let $(1 \leq i_1 < i_2 < \dots < i_k \leq n)$ and $(1 \leq j_1 < j_2 < \dots < j_k \leq n)$ be integer sequences such that for all $p \in [k]$, $\mathbf{m}_\sigma^r(i_p) = \mathbf{m}_\pi^r(j_p)$. The existence of such sequences is guaranteed by the definition of $\ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. Now for all $p \in [k]$, define $\sigma'(i_p)$ to be the smallest integer $l \in [n]$ such that $\mathbf{m}_\sigma(l) = \mathbf{m}_\sigma(i_p)$ and if $q \in [k]$ with $q < p$, then $\mathbf{m}_\sigma^r(i_q) = \mathbf{m}_\pi^r(i_p)$ implies $\sigma'(i_q) < \sigma'(i_p) = l$. For all $p \in [k]$, define $\pi(j_p)$ similarly. Then for all $p \in [k]$, $\sigma'(i_p) = \pi'(j_p)$. The remaining terms of σ' and π' may easily be chosen in such a manner that $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$. Thus there exist $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ such that $\ell(\sigma', \pi') \geq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. ■

The following example helps to illuminate the choice of σ' and π' in the proof above. If $\mathbf{m}_\sigma^r = (2, 1, 2, 1, 3, 3)$, and $\mathbf{m}_\pi^r = (3, 2, 2, 1, 3, 1)$, then we have $\ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) = 4$, with the common subsequence $(2, 2, 1, 3)$ of maximal length. Here $(1, 3, 4, 6)$ and $(2, 3, 4, 5)$ are sequences with $\mathbf{m}_\sigma^r(1) = \mathbf{m}_\pi^r(2)$, $\mathbf{m}_\sigma^r(3) = \mathbf{m}_\pi^r(3)$, $\mathbf{m}_\sigma^r(4) = \mathbf{m}_\pi^r(4)$, and $\mathbf{m}_\sigma^r(6) = \mathbf{m}_\pi^r(5)$. Then following the convention outlined in the proof above, $\sigma'(1) = \pi'(2) = 3$, $\sigma'(3) = \pi'(3) = 4$, $\sigma'(4) = \pi'(4) = 1$, and $\sigma'(6) = \pi'(5) = 5$, so that $\ell(\sigma', \pi') \geq 4$. The other elements of σ' and π' can be chosen

as follows so that $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$: set $\sigma'(2) = 1$, $\sigma'(5) = 6$, $\pi'(1) = 1$, and $\pi'(6) = 6$.

If two multipermutations \mathbf{m}_σ^r and \mathbf{m}_π^r have a common subsequence of length k , then \mathbf{m}_σ^r can be transformed into \mathbf{m}_π^r with $n-k$ (but no fewer) delete/insert operations. Delete/insert operations correspond to applying (multiplying on the right) a translocation. Hence by Lemma 1 we can state the following lemma about the r -regular Ulam distance.

Lemma 2. $d_o^r(\sigma, \pi) = \min\{k \in \mathbb{Z}_{\geq 0} : \text{there exists } (\phi_1, \phi_2, \dots, \phi_k) \text{ s.t. } \mathbf{m}_\sigma^r \cdot \phi_1 \phi_2 \dots \phi_k = \mathbf{m}_\pi^r\}.$

Proof: There exists a translocation $\phi \in \mathbb{S}_n$ such that $\ell(\mathbf{m}_\sigma^r \cdot \phi, \mathbf{m}_\pi^r) = \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) + 1$, since it is always possible to arrange one element with a single translocation. This then implies that $\min\{k \in \mathbb{Z} : \text{there exists } (\phi_1, \dots, \phi_k) \text{ s.t. } \mathbf{m}_\sigma^r \cdot \phi_1 \dots \phi_k = \mathbf{m}_\pi^r\} \leq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) = d_o^r(\sigma, \pi)$. At the same time, given $\ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) \leq n$, then for all translocations $\phi \in \mathbb{S}_n$, we have that $\ell(\mathbf{m}_\sigma^r \cdot \phi, \mathbf{m}_\pi^r) \leq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) + 1$, since a single translocation can only arrange one element at a time. Therefore $\min\{k \in \mathbb{Z} : \text{there exists } (\phi_1, \dots, \phi_k) \text{ s.t. } \mathbf{m}_\sigma^r \cdot \phi_1 \dots \phi_k = \mathbf{m}_\pi^r\} \geq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) = d_o^r(\sigma, \pi)$, by Lemma 1. ■

Lemmas 1 and 2 allow us to view the Ulam metric for r -regular multipermutations similarly to the way we view the Ulam metric for permutations; in terms of longest common subsequences or in terms of the minimum number of translocations. Another known property of the Ulam metric for permutations is left invariance, i.e. given $\tau \in \mathbb{S}_n$, we have $d_o(\sigma, \pi) = d_o(\tau\sigma, \tau\pi)$. However, left invariance does not hold in general for multipermutations, as the next lemma indicates.

Lemma 3. Let $n/r \geq 2$ and $r \geq 2$. Then there exist $\sigma', \pi' \in \mathbb{S}_n$ such that $d_o^r(e, \sigma') \neq d_o^r(\pi'e, \pi'\sigma')$.

Proof: Let $n/r \geq 2$ and $r \geq 2$. Define $\sigma', \pi' \in \mathbb{S}_n$ by

$$\sigma' := [\underbrace{2r, 2r-1, \dots, 1}_{2r}, \underbrace{2r+1, 2r+2, \dots, n}_{n-2r}] \text{ and}$$

$$\pi' := [\underbrace{1, r+1, 2, r+2, \dots, r}_{2r}, \underbrace{2r, 2r+1, 2r+2, \dots, n}_{n-2r}].$$

First, consider $d_o^r(e, \sigma')$. Note that for \mathbf{m}_e^r and $\mathbf{m}_{\sigma'}^r$, for any integer i such that $2r < i \leq n$ we have $e(i) = \sigma'(i)$, which implies $\mathbf{m}_e^r(i) = \mathbf{m}_{\sigma'}^r(i)$. Meanwhile, the first $2r$ elements of \mathbf{m}_e^r and $\mathbf{m}_{\sigma'}^r$ are $(\underbrace{1, 1, \dots, 1}_r, \underbrace{2, 2, \dots, 2}_r)$ and

$(\underbrace{2, 2, \dots, 2}_r, \underbrace{1, 1, \dots, 1}_r)$ respectively, so that the longest com-

mon subsequence of the first $2r$ elements of \mathbf{m}_e^r and $\mathbf{m}_{\sigma'}^r$ is comprised of r 1's or r 2's. Hence $\ell(\mathbf{m}_e^r, \mathbf{m}_{\sigma'}^r) = (n-2r) + r = n-r$, which by lemma 1 implies that $d_o^r(e, \sigma') = r \geq 2$.

Next, consider $d_o^r(\pi'e, \pi'\sigma')$. Multiplying π' and σ' yields

$$\pi'\sigma' = [\underbrace{2r, r, 2r-1, r-1, \dots, r+1, 1}_{2r}, \underbrace{2r+1, 2r+2, \dots, n}_{n-2r}].$$

For all integers i such that $2r < i \leq n$, we then have $\pi'e(i) =$

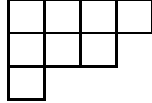
$\pi'(i) = \pi'\sigma'(i) \implies \mathbf{m}_{\pi'e}^r(i) = \mathbf{m}_{\pi'\sigma'}^r(i)$. Meanwhile, the first $2r$ elements of $\mathbf{m}_{\pi'e}^r$ and $\mathbf{m}_{\pi'\sigma'}^r$ are $(1, 2, 1, 2, \dots, 1, 2)$ and $(2, 1, 2, 1, \dots, 2, 1)$ respectively. Thus the longest common subsequence of the first $2r$ elements of $\mathbf{m}_{\pi'e}^r$ and $\mathbf{m}_{\pi'\sigma'}^r$ is any length $2r - 1$ sequence of alternating 1's and 2's. Hence $\ell(\mathbf{m}_{\pi'e}^r, \mathbf{m}_{\pi'\sigma'}^r) = (n - 2r) + (2r - 1) = n - 1$, which by lemma 1 implies that $d_o^r(\pi'e, \pi'\sigma') = 1$. ■

The fact that left invariance does not hold for the r -regular Ulam metric has implications on r -regular Ulam sphere sizes, defined and discussed in the next section. Left invariance implies sphere size does not depend upon the center. However, we will demonstrate that in the multipermutation case sizes may differ depending upon the center, a fact previously unknown.

IV. YOUNG TABLEAUX SPHERE SIZE CALCULATION

In [10], Young tableaux and the RSK-Correspondence were utilized to calculate Ulam Sphere sizes. A similar approach can be applied to r -regular Ulam spheres of arbitrary radius centered at \mathbf{m}_e^r . It is first necessary to introduce some basic notation and definitions regarding Young tableaux. Additional information on the subject can be found in [6], [12], and [14].

A *Young diagram* is a left-justified collection of cells with a (weakly) decreasing number of cells in each row below. Listing the number of cells in each row gives a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of n , where n is the total number of cells in the Young diagram. The notation $\lambda \vdash n$ indicates λ is a partition of n . Because the partition $\lambda \vdash n$ defines a unique Young diagram and vice versa, a Young diagram may be referred to by its associated partition $\lambda \vdash n$. For example, the partition $\lambda := (4, 3, 1) \vdash 8$ has the corresponding Young diagram pictured below.

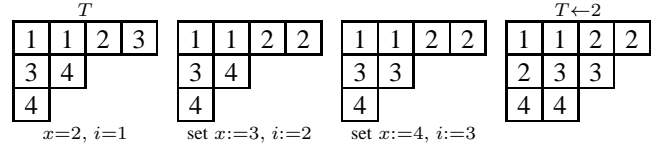


A *Young tableau*, or simply a *tableau*, is a filling of a Young diagram $\lambda \vdash n$ such that values in all cells are weakly increasing across each row and strictly increasing down each column. If each of the integers 1 through n appears exactly once in a tableau T that is a filling of a Young diagram $\lambda \vdash n$, then we call T a *standard Young tableau*, abbreviated *SYT*.

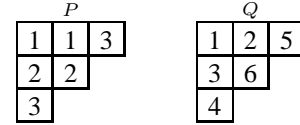
The *Schensted algorithm* is an algorithm for obtaining a tableau $T \leftarrow x$ from a tableau T and a real number x . The algorithm may be defined as follows:

- 1) Set $i := 1$.
- 2) If row i (of T) is empty or if x is greater or equal to each of the entries in the i th row then input x in a new box at the end of row i and terminate the algorithm. Otherwise, proceed to step 3.
- 3) Find the minimum entry y in row i such that $y > x$ and swap y and x . That is, replace y with x in its box and set $x := y$.
- 4) Set $i := i + 1$, and return to step 2.

As an example, let T be the tableau pictured below on the far left. The following diagrams illustrate the stages of the Schensted algorithm applied to T to obtain $T \leftarrow 2$.



The Schensted algorithm may be applied to the sequence $(\mathbf{m}_\sigma^r(1), \mathbf{m}_\sigma^r(2), \dots, \mathbf{m}_\sigma^r(n))$ of an r -regular multipermutation \mathbf{m}_σ^r to obtain a unique tableau $P := (\dots(\mathbf{m}_\sigma^r(1) \leftarrow \mathbf{m}_\sigma^r(2)) \leftarrow \dots) \leftarrow \mathbf{m}_\sigma^r(n)$. Meanwhile, a unique standard tableau results from recording where each new box appears in the construction of P . This recording is accomplished by inputting the value i in the new box that appears when $\mathbf{m}_\sigma^r(i)$ is added (via the Schendsted algorithm) to $(\dots(\mathbf{m}_\sigma^r(1) \leftarrow (\mathbf{m}_\sigma^r(2) \leftarrow \dots) \leftarrow \mathbf{m}_\sigma^r(i-1))$. Following conventions, we denote the standard tableau resulting from this recording method by Q . As an example, the two tableaux pictured below are the respective P and Q resulting from the multipermutation $(2, 3, 2, 1, 3, 1)$. Notice that P and Q have the same shape. Intermediate steps are omitted for brevity.



The *RSK-correspondence* ([6], [14]) provides a bijection between r -regular multipermutations \mathbf{m}_σ^r and ordered pairs (P, Q) on the same Young diagram $\lambda \vdash n$, where P is a tableaux whose members come from \mathbf{m}_σ^r and Q is a *SYT*. A stronger form of the following lemma appears in [6].

Lemma 4. *Let $\sigma \in S_n$ and P be the tableau resulting from running the Schendsted algorithm on the entries of σ . Then the number of columns in P is equal to $\ell(\mathbf{m}_e^r, \mathbf{m}_\sigma^r)$, the length of the longest non-decreasing subsequence of \mathbf{m}_σ^r .*

The above lemma, in conjunction with the RSK-correspondence, means that for all $k \in [n]$, the size of the set $\{\mathbf{m}_\sigma^r \in \mathcal{M}_r(S_n) : \ell(\mathbf{m}_e^r, \mathbf{m}_\sigma^r) = k\}$ is equal to the sum of the number of ordered pairs (P, Q) on each Young diagram $\lambda \vdash n$ such that $\lambda_1 = k$, where P is a tableaux whose members come from \mathbf{m}_σ^r and Q is a *SYT*. The number of *SYT* on a particular $\lambda \vdash n$ is denoted by f^λ . We denote by K_r^λ (our own notation) the number of Young tableaux on $\lambda \vdash n$ such that each $i \in [n/r]$ appears exactly r times. The next lemma states the relationship between $|S(\mathbf{m}_e^r, t)|$, f^λ , and K_r^λ .

Lemma 5. *Let $t \in [0, n-1]$, and $\Lambda := \{\lambda \vdash n : \lambda_1 \geq n-t\}$. Then $|S(\mathbf{m}_e^r, t)| = \sum_{\lambda \in \Lambda} (f^\lambda)(K_r^\lambda)$.*

Proof: Assume $t \in [0, n-1]$. Let $\Lambda := \{\lambda \vdash n : \lambda_1 \geq n-t\}$. Furthermore, let $\Lambda^{(l)} := \{\lambda \vdash n : \lambda_1 = l\}$, the set of all partitions of n having exactly l columns. By the RSK-Correspondence, and Lemma 4, there is a bijection between the set $\{\mathbf{m}_\sigma^r : \ell(\mathbf{m}_e^r, \mathbf{m}_\sigma^r) = l\}$ and the set of ordered pairs

(P, Q) where both P and Q have exactly l columns. This implies that $|\{\mathbf{m}_\sigma^r : \ell(\mathbf{m}_e^r, \mathbf{m}_\sigma^r) = l\}| = \sum_{\lambda \in \Lambda^{(l)}} (f^\lambda)(K_r^\lambda)$.

Note that by Lemma 1,

$$\begin{aligned} |S(\mathbf{m}_e^r, t)| &= |\{\mathbf{m}_\sigma : d_o^r(e, \sigma) \leq t\}| \\ &= |\{\mathbf{m}_\sigma : \ell(\mathbf{m}_e^r, \mathbf{m}_\sigma^r) \geq n - t\}|. \end{aligned}$$

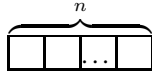
Hence it follows that $|S(\mathbf{m}_e^r, t)| = \sum_{\lambda \in \Lambda} (f^\lambda)(K_r^\lambda)$. ■

The formula below, known as the *hook length formula*, is due to Frame, Robinson, and Thrall [5], [6]. In the formula, the notation $(i, j) \in \lambda$ is used to refer to the cell in the i th row and j th column of a Young diagram $\lambda \vdash n$. The notation $h(i, j)$ denotes the *hook length* of $(i, j) \in \lambda$, i.e., the number of boxes below or to the right of (i, j) , including the box (i, j) itself. More formally, $h(i, j) := |\{(i, j^*) \in \lambda : j^* \geq j\} \cup \{(i^*, j) \in \lambda : i^* \geq i\}|$. The formula is as follows:

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h(i, j)}.$$

Thus by applying Lemma 5, it is possible to calculate r -regular sphere size by using the hook length formula. We will use this strategy to treat the sphere of radius $r = 1$. However, because sphere sizes are calculated recursively, we must first calculate the sphere size when $r = 0$.

Remark. $|S(\mathbf{m}_e^r, 0)| = 1$. Although this is an obvious fact, we wish to consider why it is true from the perspective of Lemma 5. Note first that there is only one partition $\lambda \vdash n$ such that $\lambda_1 = n$, namely $\lambda' := (n)$ with the associated Young diagram below.

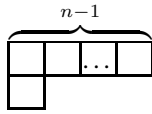


It is clear that there is only one possible Young tableau on λ' so that $(f^{\lambda'}) = 1$, and thus by Lemma 5 $|S(\mathbf{m}_e^r, 0)| = 1$.

The following proposition is an application of Lemma 5.

Proposition 6. $|S(\mathbf{m}_e^r, 1)| = 1 + (n - 1)(n/r - 1)$.

Proof: First note that $|S(\mathbf{m}_e^r, 0)| = 1$. There is only one possible partition $\lambda \vdash n$ such that $\lambda_1 = n - 1$, namely $\lambda := (n - 1, 1)$, with its Young diagram pictured below.



Therefore by Lemma 5, $|S(\mathbf{m}_e^r, 1)| = 1 + (f^{\lambda'})(K_r^{\lambda'})$. Applying the well-known hook length formula ([5], [6]), we obtain $f^{\lambda'} = n - 1$. The value $K_r^{\lambda'}$ is characterized by possible fillings of row 2 with the stipulation that each $i \in [n/r]$ must appear exactly r times in the diagram. In this case, since there is only a single box in row 2, the possible fillings are $i \in [n/r - 1]$, each of which yields a unique Young tableau of the desired type. Hence $K_r^{\lambda'} = n/r - 1$, which implies that $|S(\mathbf{m}_e^r, 1)| = 1 + (n - 1)(n/r - 1)$. ■

Proposition 6 demonstrates how Young Tableaux may be used to calculate r -regular Ulam spheres centered at \mathbf{m}_e^r .

V. r -REGULAR ULAM SPHERES AND DUPLICATION SETS

In the previous section we showed how multipermutation Ulam spheres may be calculated when the center is \mathbf{m}_e^r . In this section we provide a way to calculate sphere sizes for any center when the radius is $t = 1$. The r -regular Ulam sphere sizes play an important role in understanding the potential code size for a given minimum distance.

For example, the well-known sphere-packing and Gilbert-Varshamov bounds rely on calculating, or at least bounding sphere sizes. In the case of permutations, recall that the Ulam sphere $S(\sigma, t)$ centered at σ of radius t was defined as $S(\sigma, t) := \{\pi \in \mathbb{S}_n : d_o(\sigma, \pi) \leq t\}$, which is equivalent by definition to the set $\{\pi \in \mathbb{S}_n : n - \ell(\sigma, \pi) \leq t\}$.

In the case of r -regular multipermutations, for $t \in \mathbb{Z}_{>0}$, we introduce the following analogous definition of a sphere.

Definition. Define

$$S(\mathbf{m}_\sigma^r, t) := \{\mathbf{m}_\pi^r \in \mathcal{M}_r(\mathbb{S}_n) : d_o^r(\sigma, \pi) \leq t\}$$

We call $S(\mathbf{m}_\sigma^r, t)$ the *r -regular Ulam sphere* centered at \mathbf{m}_σ^r of radius t .

By Lemma 1, $S(\mathbf{m}_\sigma^r, t) = \{\mathbf{m}_\pi^r \in \mathcal{M}_r(\mathbb{S}_n) : n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) \leq t\}$.

It should be noted, however, that the notation \mathbf{m}_π^r is a bit misleading because given $\mathbf{m}_\pi^r \in \mathcal{M}(\mathbb{S}_n)$, we cannot uniquely determine π . The r -regular Ulam sphere definition can also be viewed in terms of translocations. Lemma 2 implies that $S(\mathbf{m}_\sigma^r, t)$ is equivalent to $\{\mathbf{m}_\pi^r \in \mathcal{M}_r(\mathbb{S}_n) : \text{there exists } k \in [t] \text{ and } (\phi_1, \dots, \phi_k) \text{ s.t. } \mathbf{m}_\sigma^r \cdot \phi_1 \cdots \phi_k = \mathbf{m}_\pi^r\}$.

Lemma 5 provided a way to calculate r -regular Ulam spheres centered at \mathbf{m}_e^r . Unfortunately, the choice of center has an impact on the size of the sphere, as is easily confirmed by comparing Proposition 6 to Lemma 16 (Section VI). Hence the applicability of Lemma 5 is limited.

We begin to address the issue of differing sphere sizes by considering the radius $t = 1$ case. To aid with calculating such sphere sizes, we introduce (as our own definition) the following subset of the set of translocations.

Definition. Let $n \in \mathbb{Z}_{>0}$. Define

$$T_n := \{\phi(i, j) \in \mathbb{S}_n : i - j \neq 1\}.$$

We call T_n the *unique set of translocations*.

By definition, T_n is the set of all translocations in \mathbb{S}_n , except translocations of the form $\phi(i, i - 1)$. We exclude translocations of this form because they can be modeled by translocations of the form $\phi(i - 1, i)$, and are therefore redundant.

We claim that the set T_n is precisely the set of translocations needed to obtain all unique permutations within the Ulam sphere of radius 1 via multiplication. Moreover, there is no redundancy in the set, that is, there is no smaller set of translocations yielding the entire Ulam sphere of radius 1 when

multiplied with a given center permutation. These facts are stated in the next lemma.

Lemma 7. $S(\sigma, 1) = \{\sigma\phi \in \mathbb{S}_n : \phi \in T_n\}$ and $|T_n| = |S(\sigma, 1)|$.

Proof: We will first show that $S(\sigma, 1) = \{\sigma\phi \in \mathbb{S}_n : \phi \in T_n\}$. Note that

$$\begin{aligned} S(\sigma, 1) &= \{\pi \in \mathbb{S}_n : d_o(\sigma, \pi) \leq 1\} \\ &= \{\sigma\phi(i, j) \in \mathbb{S}_n : i, j \in [n]\}. \end{aligned}$$

It is trivial that

$$\begin{aligned} T_n &= \{\phi(i, j) \in \mathbb{S}_n : i - j \neq 1\} \\ &\subseteq \{\phi(i, j) \in \mathbb{S}_n : i, j \in [n]\}. \end{aligned}$$

Therefore $\{\sigma\phi \in \mathbb{S}_n : \phi \in T_n\} \subseteq S(\sigma, 1)$.

To see why $S(\sigma, 1) \subseteq \{\sigma\phi \in \mathbb{S}_n : \phi \in T_n\}$, consider any $\sigma\phi(i, j) \in \{\sigma\phi(i, j) \in \mathbb{S}_n : i, j \in [n]\} = S(\sigma, 1)$. If $i - j \neq 1$, then $\phi(i, j) \in T_n$, and thus $\sigma\phi(i, j) \in \{\sigma\phi \in \mathbb{S}_n : \phi \in T_n\}$. Otherwise, if $i - j = 1$, then $\sigma\phi(i, j) = \sigma\phi(j, i)$, and $i - j = 1 \implies j - i = -1 \neq 1$, so $\phi(j, i) \in T_n$. Hence $\sigma\phi(i, j) = \sigma\phi(j, i) \in \{\sigma\phi \in \mathbb{S}_n : \phi \in T_n\}$.

Next we show that $|T_n| = |S(\sigma, 1)|$. By Proposition 6 (in the case that $r = 1$), $|S(\sigma, 1)| = 1 + (n - 1)^2$. On the other hand, $|T_n| = |\{\phi(i, j) \in \mathbb{S}_n : i - j \neq 1\}|$. If $i = 1$, then there are n values $j \in [n]$ such that $i - j \neq 1$. Otherwise, if $i \in [n]$ but $i \neq 1$, then there are $n - 1$ values $j \in [n]$ such that $i - j \neq 1$. However, for all $i, j \in [n]$, $\phi(i, i) = \phi(j, j) = e$ so that there are $n - 1$ redundancies. Therefore $|T_n| = n + (n - 1)(n - 1) - (n - 1) = 1 + (n - 1)^2$. ■

In the case of permutations, the set T_n has no redundancies. If $\phi_1, \phi_2 \in T_n$, then $\sigma\phi_1 = \sigma\phi_2$ implies $\phi_1 = \phi_2$. Alternatively, in the case of multipermutations, the set T_n can generally be shrunk to exclude redundancies. Notice that $S(\mathbf{m}_\sigma^r, 1) = \{\mathbf{m}_\pi^r \in \mathcal{M}_r(\mathbb{S}_n) : \text{there exists } \phi \text{ s.t. } \mathbf{m}_\sigma^r \cdot \phi = \mathbf{m}_\pi^r\}$, which is equal to $\{\mathbf{m}_\sigma^r \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) : \phi \in T_n\}$. However, it is possible that there exist $\phi_1, \phi_2 \in T_n$ such that $\phi_1 \neq \phi_2$, but $\mathbf{m}_\sigma^r \cdot \phi_1 = \mathbf{m}_\sigma^r \cdot \phi_2$. In such an instance we may refer to either ϕ_1 or ϕ_2 as a **duplicate translocation** for \mathbf{m}_σ^r .

If we remove all duplicate translocations for \mathbf{m}_σ^r from T_n , then the resulting set will have the same cardinality as the r -regular Ulam sphere of radius 1 centered at \mathbf{m}_σ^r . The next definition (our own) is the set of standard duplicate translocations. For the remainder of the paper, assume that \mathbf{m} is an n -length integer tuple, i.e. $\mathbf{m} \in \mathbb{Z}^n$.

Definition. Define

$$\begin{aligned} D(\mathbf{m}) &:= \{ \phi(i, j) \in T_n \setminus \{e\} : (\mathbf{m}(i) = \mathbf{m}(j)) \\ &\quad \text{or } (\mathbf{m}(i) = \mathbf{m}(i - 1) \text{ or } i = 1) \} \end{aligned}$$

We call $D(\mathbf{m})$ the **standard duplicate translocation set** for \mathbf{m} . For each $i \in [n]$, also define $D_i(\mathbf{m}) := \{\phi(i, j) \in D_n : j \in [n]\}$.

If we take an r -regular multipermutation \mathbf{m}_σ^r , then removing $D(\mathbf{m}_\sigma^r)$ from T_n equates to removing a set of duplicate translocations. These duplications come in two varieties. The

first variety corresponds to the first condition of the $D(\mathbf{m})$ definition, when $\mathbf{m}(i) = \mathbf{m}(j)$. For example, if $\sigma \in \mathbb{S}_6$ such that $\mathbf{m}_\sigma^2 = (1, 3, 2, 2, 3, 1)$, then we have $\mathbf{m}_\sigma^2 \cdot \phi(1, 5) = (3, 2, 2, 3, 1, 1) = \mathbf{m}_\sigma^2 \cdot \phi(1, 6)$, since $\mathbf{m}_\sigma^2(2) = 3 = \mathbf{m}_\sigma^2(4)$. This is because moving the first 1 to the left or to the right of the last 1 results in the same tuple.

The second variety corresponds to the second condition of the $D(\mathbf{m})$ definition above, when $\mathbf{m}(i) = \mathbf{m}(i - 1)$. For example, if $\mathbf{m}_\sigma^2 = (1, 3, 2, 2, 3, 1)$ as before, then for all $j \in \{1, 2, 3, 4, 5, 6\}$, we have $\mathbf{m}_\sigma^2 \cdot \phi(3, j) = \mathbf{m}_\sigma^2 \cdot \phi(4, j)$. This is because any translocation that deletes and inserts the second of the two adjacent 2's does not result in a different tuple when compared to deleting and inserting the first of the two adjacent 2's.

Lemma 8. $S(\mathbf{m}_\sigma^r, 1) = \{\mathbf{m}_\sigma^r \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) : \phi \in T_n \setminus D(\mathbf{m}_\sigma^r)\}$.

Proof: Notice $S(\mathbf{m}_\sigma^r, 1) = \{\mathbf{m}_\sigma^r \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) : \phi \in T_n\}$. Hence it suffices to show that for all $\phi(i, j) \in D(\mathbf{m}_\sigma^r)$, there exists some $i', j' \in [n]$ such that $\phi(i', j') \in T_n \setminus D(\mathbf{m}_\sigma^r)$ and $\mathbf{m}_\sigma^r \cdot \phi(i, j) = \mathbf{m}_\sigma^r \cdot \phi(i', j')$. We proceed by dividing the proof into two main cases. Case I is when $(\mathbf{m}_\sigma^r(i) \neq \mathbf{m}_\sigma^r(i - 1))$ or $i = 1$. Case II is when $(\mathbf{m}_\sigma^r(i) = \mathbf{m}_\sigma^r(i - 1))$.

Case I (when $(\mathbf{m}_\sigma^r(i) \neq \mathbf{m}_\sigma^r(i - 1))$ or $i = 1$) can be split into two subcases:

Case IA: $i < j$

Case IB: $i > j$.

We can ignore the instance when $i = j$, since $\phi(i, j) \in D(\mathbf{m}_\sigma^r)$ implies $i \neq j$. For case IA, if for all $p \in [i, j]$ (for $a, b \in \mathbb{Z}$ with $a < b$, the notation $[a, b] := \{a, a + 1, \dots, b\}$) we have $\mathbf{m}_\sigma^r(i) = \mathbf{m}_\sigma^r(p)$, then $\mathbf{m}_\sigma^r \cdot \phi(i, j) = \mathbf{m}_\sigma^r \cdot e$. Thus setting $i' = j' = 1$ yields the desired result. Otherwise, if there exists $p \in [i, j]$ such that $\mathbf{m}_\sigma^r(i) \neq \mathbf{m}_\sigma^r(p)$, then let

$$j^* := j - \min\{k \in \mathbb{Z}_{>0} : \mathbf{m}_\sigma^r(i) \neq \mathbf{m}_\sigma^r(j - k)\}.$$

Then $\phi(i, j^*) \in T_n \setminus D(\mathbf{m}_\sigma^r)$ and $\mathbf{m}_\sigma^r \cdot \phi(i, j) = \mathbf{m}_\sigma^r \cdot \phi(i, j^*)$. Thus setting $i' = i$ and $j' = j^*$ yields the desired result. Case IB is similar to Case IA.

Case II (when $\mathbf{m}_\sigma^r(i) = \mathbf{m}_\sigma^r(i - 1)$), can also be divided into two subcases.

Case IIA: $i < j$

Case IIB: $i > j$.

As in Case I, we can ignore the instance when $i = j$. For Case IIA, if for all $p \in [i, j]$ we have $\mathbf{m}_\sigma^r(i) = \mathbf{m}_\sigma^r(p)$, then $\mathbf{m}_\sigma^r \cdot \phi(i, j) = \mathbf{m}_\sigma^r \cdot e$, so setting $i = j = 1$ achieves the desired result. Otherwise, if there exists $p \in [i, j]$ such that $\mathbf{m}_\sigma^r(i) \neq \mathbf{m}_\sigma^r(p)$, then let

$$i^* := i - \min\{k \in \mathbb{Z}_{>0} : (\mathbf{m}_\sigma^r(i) \neq \mathbf{m}_\sigma^r(i - k - 1)) \text{ or } (i - k = 1)\}.$$

Then $\mathbf{m}_\sigma^r \cdot \phi(i, j) = \mathbf{m}_\sigma^r \cdot \phi(i^*, j)$ and either one of the following is true: (1) $\phi(i^*, j) \notin D_i(\mathbf{m}_\sigma^r) \implies \phi(i^*, j) \notin D(\mathbf{m}_\sigma^r)$, so set $i' = i^*$ and $j' = j$; or (2) by Case IA there exist $i', j' \in [n]$ such that $\phi(i', j') \in T_n \setminus D(\mathbf{m}_\sigma^r)$ and $\mathbf{m}_\sigma^r \cdot \phi(i', j') = \mathbf{m}_\sigma^r \cdot \phi(i^*, j) = \mathbf{m}_\sigma^r \cdot \phi(i, j)$. Case IIB is

similar to Case IIA. \blacksquare

While Lemma 8 shows that $D(\mathbf{m}_\sigma^r)$ is a set of duplicate translocations for \mathbf{m}_σ^r , we have not shown that $T_n \setminus D(\mathbf{m}_\sigma^r)$ is the set of minimal size having the quality that $S(\mathbf{m}_\sigma^r, 1) = \{\mathbf{m}_\sigma^r \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) : \phi \in T_n \setminus D(\mathbf{m}_\sigma^r)\}$. In fact it is not minimal. In some instances it is possible to remove further duplicate translocations to reduce the set size. We will define another set of duplicate translocations, but a few preliminary definitions are first necessary.

We say that \mathbf{m} is *alternating* if for all odd integers $1 \leq i \leq n$, $\mathbf{m}(i) = \mathbf{m}(1)$ and for all even integers $2 \leq i' \leq n$, $\mathbf{m}(i') = \mathbf{m}(2)$ but $\mathbf{m}(1) \neq \mathbf{m}(2)$. In other words, any alternating tuple is of the form $(a, b, a, b, \dots, a, b)$ or (a, b, a, b, \dots, a) where $a, b \in \mathbb{Z}$ and $a \neq b$. Any singleton is also said to be alternating. Now for integers $1 \leq i \leq n$ and $0 \leq k \leq n - i$, the *substring* $\mathbf{m}[i, i+k]$ of \mathbf{m} is defined as $\mathbf{m}[i, i+k] := (\mathbf{m}(i), \mathbf{m}(i+1), \dots, \mathbf{m}(i+k))$. Given a substring $\mathbf{m}[i, j]$ of \mathbf{m} , the *length* $|\mathbf{m}[i, j]|$ of $\mathbf{m}[i, j]$ is defined as $|\mathbf{m}[i, j]| := j - i + 1$. As an example, if $\mathbf{m}' := (1, 2, 2, 4, 2, 4, 3, 1, 3)$, then $\mathbf{m}'[3, 6] = (2, 4, 2, 4)$ is an alternating substring of \mathbf{m}' of length 4.

Definition. Next define

$$E(\mathbf{m}) := \left\{ \begin{array}{l} \phi(i, j) \in T_n \setminus D(\mathbf{m}) : \\ i < j \text{ and there exists } k \in [i, j-2] \text{ s.t.} \\ (\phi(j, k) \in T_n \setminus D(\mathbf{m})) \\ \text{and } (\mathbf{m} \cdot \phi(i, j) = \mathbf{m} \cdot \phi(j, k)) \end{array} \right\}.$$

We call $E(\mathbf{m})$ the *alternating duplicate translocation set* for \mathbf{m} because it is only nonempty when \mathbf{m} contains an alternating substring of length at least 4. For each $i \in [n]$, also define $E_i(\mathbf{m}) := \{\phi(i, j) \in E(\mathbf{m}) : j \in [n]\}$.

In the example of $\mathbf{m}' := (1, 2, 2, 4, 2, 4, 3, 1, 3)$ above, $\mathbf{m}^* \cdot \phi(2, 6) = \mathbf{m}' \cdot \phi(6, 3)$ and $\phi(2, 6), \phi(6, 3) \in T_9 \setminus D(\mathbf{m}')$, implying that $\phi(2, 6) \in E(\mathbf{m}')$. In fact, it can easily be shown that $E(\mathbf{m}') = \{\phi(2, 6)\}$.

Lemma 9. Let $i \in [n]$. Then $E_i(\mathbf{m}) \neq \emptyset$ if and only if

- 1) $\mathbf{m}(i) \neq \mathbf{m}(i-1)$
- 2) There exists $j \in [i+1, n]$ and $k \in [i, j-2]$ such that
 - i) For all $p \in [i, k-1]$, $\mathbf{m}(p) = \mathbf{m}(p+1)$
 - ii) $\mathbf{m}[k, j]$ is alternating
 - iii) $|\mathbf{m}[k, j]| \geq 4$.

Proof: Let $i \in [n]$. We will first assume 1) and 2) in the lemma statement and show that $E_i(\mathbf{m})$ is not empty. Suppose $\mathbf{m}(i) \neq \mathbf{m}(i-1)$, and that there exists $j \in [i+1, n]$ and $k \in [i, j-2]$ such that for all $p \in [i, k-1]$, we have $\mathbf{m}(p) = \mathbf{m}(p+1)$. Suppose also that $\mathbf{m}[k, j]$ is alternating with $|\mathbf{m}[k, j]| \geq 4$.

For ease of notation, let $a := \mathbf{m}(k) = \mathbf{m}(k+2)$ and $b := \mathbf{m}(k+1) = \mathbf{m}(k+3)$ so that $\mathbf{m}[k, k+3] = (a, b, a, b) \in \mathbb{Z}^4$. Then

$$\begin{aligned} (\mathbf{m} \cdot \phi(i, k+3))[k, k+3] &= (\mathbf{m} \cdot \phi(k, k+3))[k, k+3] \\ &= (b, a, b, a) \\ &= (\mathbf{m} \cdot \phi(k+3, k))[k, k+3]. \end{aligned}$$

Moreover, for all $p \notin [k, k+3]$, we have $(\mathbf{m} \cdot \phi(i, k+3))(p) = \mathbf{m}(p) = (\mathbf{m} \cdot \phi(k+3, k))(p)$. Therefore $\mathbf{m} \cdot \phi(i, k+3) = \mathbf{m} \cdot \phi(k+3, k)$. Also notice that $\mathbf{m}(i) \neq \mathbf{m}(i-1)$ implies that $\mathbf{m} \cdot \phi(i, k+3) \notin D(\mathbf{m})$. Hence $\phi(i, k+3) \in E_i(\mathbf{m})$.

We now prove the second half of the lemma. That is, we assume that $E_i(\mathbf{m}) \neq \emptyset$ and then show that 1) and 2) necessarily hold. Suppose that $E_i(\mathbf{m})$ is nonempty. Then $\mathbf{m}(i) \neq \mathbf{m}(i-1)$, since otherwise there would not exist any $\phi(i, j) \in T_n \setminus D(\mathbf{m})$.

Let $j \in [i+1, n]$ and $k \in [i, j-2]$ such that $\phi(j, k) \in T_n \setminus D(\mathbf{m})$ and $\mathbf{m} \cdot \phi(i, j) = \mathbf{m}(j, k)$. Existence of such j, k , and $\phi(j, k)$ is guaranteed by definition of $E_i(\mathbf{m})$ and the fact that $E_i(\mathbf{m})$ was assumed to be nonempty. Then for all $p \in [i, k-1]$, we have $\mathbf{m}(p) = \mathbf{m}(p+1)$ and for all $p \in [k, j-2]$, we have $\mathbf{m}(p) = \mathbf{m}(p+2)$. Hence either $\mathbf{m}[k, j]$ is alternating, or else for all $p, q \in [k, j]$, we have $\mathbf{m}(p) = \mathbf{m}(q)$. However, the latter case is impossible, since it would imply that for all $p, q \in [i, j]$ that $\mathbf{m}(p) = \mathbf{m}(q)$, which would mean $\phi(j, k) \notin T_n \setminus D(\mathbf{m})$, a contradiction. Therefore $\mathbf{m}[k, j]$ is alternating.

It remains only to show that $|\mathbf{m}[k, j]| \geq 4$. Since $k \in [i, j-2]$, it must be the case that $|\mathbf{m}[k, j]| \geq 3$. However, if $|\mathbf{m}[k, j]| = 3$ (which occurs when $k = j-2$), then $(\mathbf{m} \cdot \phi(i, j))(j) = \mathbf{m}(i) = \mathbf{m}(k) \neq \mathbf{m}(k+1) = (\mathbf{m} \cdot \phi(j, k))(j)$, which implies that $\mathbf{m} \cdot \phi(i, j) \neq \mathbf{m} \cdot \phi(j, k)$, a contradiction. Hence $|\mathbf{m}[k, j]| \geq 4$. \blacksquare

One implication of Lemma 9 is that there are only two possible forms for $\mathbf{m}[i, j]$ where $\phi(i, j) \in E_i(\mathbf{m})$. The first possibility is that $\mathbf{m}[i, j]$ is an alternating substring of the form $(a, b, a, b, \dots, a, b)$ (here $a, b \in \mathbb{Z}$), so that $\mathbf{m}[i, j] \cdot \phi(i, j)$ is of the form $(b, a, b, a, \dots, b, a)$. In this case, as long as $|\mathbf{m}[i, j]| \geq 4$, then setting $k = i$ implies that $k \in [i, j-2]$, that $\phi(j, k) \in T_n \setminus D(\mathbf{m})$, and that $\mathbf{m}[i, j] \cdot \phi(i, j) = \mathbf{m}[i, j] \cdot \phi(j, k)$.

The other possibility is that $\mathbf{m}[i, j]$ is of the form $(\underbrace{a, a, a, \dots, a}_k, \underbrace{b, a, b, \dots, a}_{n-k})$ (again $a, b \in \mathbb{Z}$), so that $\mathbf{m}[i, j] \cdot \phi(i, j)$ is of the form $(\underbrace{a, \dots, a}_{k-1}, \underbrace{b, a, b, \dots, b}_{n-k+1})$. Again in this case, as long as $|\mathbf{m}[i, j]| \geq 4$, then $k \in [i, j-2]$, $\phi(j, k) \in T_n \setminus D(\mathbf{m})$, and $\mathbf{m}[i, j] \cdot \phi(i, j) = \mathbf{m}[i, j] \cdot \phi(j, k)$.

Remark. If

- 1) \mathbf{m} is alternating and
 - 2) n is even
- then $\mathbf{m} \cdot \phi(1, n) = \mathbf{m} \cdot \phi(n, 1)$.

Remark. If

- 1) \mathbf{m} is alternating
 - 2) $n \geq 3$
 - 3) n is odd,
- then $\mathbf{m} \cdot \phi(1, n) \neq \mathbf{m} \cdot \phi(n, 1)$.

To calculate $|E(\mathbf{m}_\sigma^r)|$, we define a set of equal size that is easier to calculate.

Definition. Define

$$E^*(\mathbf{m}) := \{ (i, j) \in [n] \times [n] : \begin{array}{l} (\mathbf{m}[i, j] \text{ is alternating}) \\ \text{and } (|\mathbf{m}[i, j]| \geq 4) \\ \text{and } (|\mathbf{m}[i, j]| \text{ is even}) \end{array} \}.$$

For each $i \in [n]$, also define $E_i^*(\mathbf{m}) := \{(i, j) \in E^*(\mathbf{m}) : j \in [n]\}$. Notice that $E^*(\mathbf{m}) = \bigcup_{i \in [n]} E_i^*(\mathbf{m})$.

Lemma 10. $|E(\mathbf{m})| = |E^*(\mathbf{m})|$

Proof: The idea of the proof is simple. Each element $\phi(i, j) \in E(\mathbf{m})$ involves exactly one alternating sequence of length greater or equal to 4, so the set sizes must be equal. We formalize the argument by showing that $|E(\mathbf{m})| \leq |E^*(\mathbf{m})|$ and then that $|E^*(\mathbf{m})| \leq |E(\mathbf{m})|$.

To see why $|E(\mathbf{m})| \leq |E^*(\mathbf{m})|$, we define a mapping $f : [n] \rightarrow [n]$, which maps index values either to the beginning of the nearest alternating subsequence to the right, or else to n . For all $i \in [n]$, let

$$f(i) := \begin{cases} i + \min\{p \in \mathbb{Z}_{\geq 0} : (\mathbf{m}(i) \neq \mathbf{m}(i+p+1)) \vee (i+p = n)\} & \text{(if } \mathbf{m}(i) \neq \mathbf{m}(i-1) \text{ or } i = 1) \\ n & \text{(otherwise)} \end{cases}$$

Notice by definition of f , if $i, i' \in [n]$ such that $i \neq i'$, and if $\mathbf{m}(i) \neq \mathbf{m}(i-1)$ or $i = 1$ and at the same time $\mathbf{m}(i') \neq \mathbf{m}(i'-1)$ or $i' = 1$, then $f(i) \neq f(i')$.

Now for each $i \in [n]$, if $\mathbf{m}(i) \neq \mathbf{m}(i-1)$ or $i = 1$, then $|E_i(\mathbf{m})| = |E_{f(i)}^*(\mathbf{m})|$ by Lemma 9 and the two previous remarks. Otherwise, if $\mathbf{m}(i) = \mathbf{m}(i-1)$, then $|E_i(\mathbf{m})| = |E_{f(i)}^*(\mathbf{m})| = 0$. Therefore $|E_i(\mathbf{m})| \leq |E_i^*(\mathbf{m})|$. This is true for all $i \in [n]$, so $|E(\mathbf{m})| \leq |E^*(\mathbf{m})|$.

The argument to show that $|E^*(\mathbf{m})| \leq |E(\mathbf{m})|$ is similar, except it uses the following function $g : [n] \rightarrow [n]$ instead of f . For all $i \in [n]$, let

$$g(i) := \begin{cases} i - \min\{p \in \mathbb{Z}_{\geq 0} : (\mathbf{m}(i) \neq \mathbf{m}(i-p-1)) \vee (i-p = 1)\} & \text{(if } \mathbf{m}(i) \neq \mathbf{m}(i-1) \text{ or } i = n) \\ n & \text{(otherwise)} \end{cases}$$

By definition, calculating $|E^*(\mathbf{m})|$ equates to calculating the number of alternating substrings $\mathbf{m}[i, j]$ of \mathbf{m} such that the length of the substring is both even and longer than 4. The following lemma helps to simplify this calculation further.

Lemma 11. Let \mathbf{m} be an alternating string. Then

- 1) If n is even then $|E^*(\mathbf{m})| = \left(\frac{n-2}{2}\right)^2$
- 2) If n is odd then $|E^*(\mathbf{m})| = \left(\frac{n-3}{2}\right) \left(\frac{n-1}{2}\right)$

Proof: Assume \mathbf{m} is an alternating string. By Lemma 10, $|E(\mathbf{m})| = |E^*(\mathbf{m})| = |\bigcup_{i \in [n]} E_i^*(\mathbf{m})|$. Since \mathbf{m} was assumed

to be alternating,

$$\begin{aligned} & \left| \bigcup_{i \in [n]} E_i^*(\mathbf{m}) \right| \\ &= |\{(i, j) \in [n] \times [n] : |\mathbf{m}[i, j]| \geq 4 \text{ and } |\mathbf{m}[i, j]| \text{ is even}\}| \\ &= |\{(i, j) \in [n] \times [n] : j - i + 1 \in K\}|, \end{aligned}$$

where K is the set of even integers between 4 and n , i.e. $K := \{k \in [4, n] : k \text{ is even}\}$. For each $k \in K$, we have

$$\begin{aligned} & |\{(i, j) \in [n] \times [n] : j - i + 1 = k\}| \\ &= |\{i \in [n] : i \in [1, n - k + 1]\}| \\ &= n - k + 1. \end{aligned}$$

Therefore $|E(\mathbf{m})| = \sum_{k \in K} (n - k + 1)$. In the case that n is even, then

$$\begin{aligned} \sum_{k \in K} (n - k + 1) &= \sum_{i=2}^{n/2} (n - 2i + 1) \\ &= \sum_{i=1}^{(n-2)/2} (2i - 1) = \left(\frac{n-2}{2}\right)^2. \end{aligned}$$

In the case that n is odd, then

$$\begin{aligned} \sum_{k \in K} (n - k + 1) &= \sum_{i=2}^{(n-1)/2} (n - 2i + 1) \\ &= \sum_{i=1}^{(n-3)/2} 2i = \left(\frac{n-3}{2}\right) \left(\frac{n-1}{2}\right). \end{aligned}$$

Notice that by Lemma 11, it suffices to calculate $|E(\mathbf{m})|$ for locally maximal length alternating substrings of \mathbf{m} . An alternating substring $\mathbf{m}[i, j]$ is of *locally maximal length* if and only if 1) $\mathbf{m}[i-1]$ is not alternating or $i = 1$; and 2) $\mathbf{m}[i, j+1]$ is not alternating or $j = n$.

Finally, we define the general set of duplications. The lemma that follows the definition also shows that removing the set $D^*(\mathbf{m}_\sigma^r)$ from T_n removes all duplicate translocations associated with \mathbf{m}_σ^r .

Definition ($D^*(\mathbf{m})$, duplication set). Define

$$D^*(\mathbf{m}) := D(\mathbf{m}) \cup E(\mathbf{m}).$$

We call $D^*(\mathbf{m})$ the **duplication set** for \mathbf{m} . For each $i \in [n]$, we also define $D_i^*(\mathbf{m}) := \{\phi(i, j) \in D^*(\mathbf{m}) : j \in [n]\}$.

Lemma 12. Let $\phi_1, \phi_2 \in T_n \setminus D^*(\mathbf{m}_\sigma^r)$. Then $\phi_1 = \phi_2$ if and only if $\mathbf{m}_\sigma^r \cdot \phi_1 = \mathbf{m}_\sigma^r \cdot \phi_2$.

Proof: Let $\phi_1, \phi_2 \in T_n \setminus D^*(\mathbf{m}_\sigma^r)$. If $\phi_1 = \phi_2$ then $\mathbf{m}_\sigma^r \cdot \phi_1 = \mathbf{m}_\sigma^r \cdot \phi_2$ trivially. It remains to prove that $\mathbf{m}_\sigma^r \cdot \phi_1 = \mathbf{m}_\sigma^r \cdot \phi_2 \implies \phi_1 = \phi_2$. We proceed by contrapositive. Suppose that $\phi_1 \neq \phi_2$. We want to show that $\mathbf{m}_\sigma^r \cdot \phi_1 \neq \mathbf{m}_\sigma^r \cdot \phi_2$. Let $\phi_1 := \phi(i_1, j_1)$ and $\phi_2 := \phi(i_2, j_2)$. The remainder of the proof can be split into two main cases: Case I is if $i_1 = i_2$ and Case II is if $i_1 \neq i_2$.

Case I (when $i_1 = i_2$), can be further divided into two

subcases:

Case IA: $\mathbf{m}_\sigma^r(i_1) = \mathbf{m}_\sigma^r(i_1 - 1)$

Case IB: $\mathbf{m}_\sigma^r(i_1) \neq \mathbf{m}_\sigma^r(i_1 - 1)$.

Case IA is easy to prove. We have $D_{i_1}^*(\mathbf{m}_\sigma^r) = D_{i_2}^*(\mathbf{m}_\sigma^r) = \{\phi(i_1, j) \in T_n \setminus \{e\} : j \in [n]\}$, so $\phi_1 = e = \phi_2$, a contradiction. For Case IB, we can first assume without loss of generality that $j_1 < j_2$ and then split into the following smaller subcases:

- i) $(j_1 < i_1)$ and $(j_2 > i_1)$
- ii) $(j_1 < i_1)$ and $(j_2 \leq i_1)$
- iii) $(j_1 > i_1)$ and $(j_2 > i_1)$
- iv) $(j_1 > i_1)$ and $(j_2 \leq i_1)$.

However, subcase iv) is unnecessary since it was assumed that $j_1 < j_2$, so $j_1 > i_1 \implies j_2 > j_1 > i_1$. Subcase ii) can also be reduced to $(j_1 < i_1)$ and $(j_2 < i_1)$ since $j_2 \neq i_2 = i_1$. Each of the remaining subcases is proven by noting that there is some element in the multipermutation $\mathbf{m}_\sigma^r \cdot \phi_1$ that is necessarily different from $\mathbf{m}_\sigma^r \cdot \phi_2$. For example, in subcase i), we have $\mathbf{m}_\sigma^r \cdot \phi_1(j_1) = \mathbf{m}_\sigma^r(i_1) \neq \mathbf{m}_\sigma^r(j_1) = \mathbf{m}_\sigma^r \cdot \phi_2(j_1)$. Subcases ii) and iii) are solved similarly.

Case II (when $i_1 \neq i_2$) can be divided into three subcases:

Case IIA: $(\mathbf{m}_\sigma^r(i_1) = \mathbf{m}_\sigma^r(i_1 - 1) \text{ and } \mathbf{m}_\sigma^r(i_2) = \mathbf{m}_\sigma^r(i_2 - 1))$,

Case IIB: either

$(\mathbf{m}_\sigma^r(i_1) = \mathbf{m}_\sigma^r(i_1 - 1) \text{ and } \mathbf{m}_\sigma^r(i_2) \neq \mathbf{m}_\sigma^r(i_2 - 1))$

or $(\mathbf{m}_\sigma^r(i_1) \neq \mathbf{m}_\sigma^r(i_1 - 1) \text{ and } \mathbf{m}_\sigma^r(i_2) = \mathbf{m}_\sigma^r(i_2 - 1))$,

Case IIC: $(\mathbf{m}_\sigma^r(i_1) \neq \mathbf{m}_\sigma^r(i_1 - 1) \text{ and } \mathbf{m}_\sigma^r(i_2) \neq \mathbf{m}_\sigma^r(i_2 - 1))$.

Case IIA is easily solved by mimicking the proof of Case IA. Case IIB is also easily solved as follows. First, without loss of generality, we assume that $\mathbf{m}_\sigma^r(i_1) = \mathbf{m}_\sigma^r(i_1 - 1)$ and $\mathbf{m}_\sigma^r(i_2) \neq \mathbf{m}_\sigma^r(i_2 - 1)$. Then $D_{i_1}^*(\mathbf{m}_\sigma^r) = \{\phi(i_1, j) \in T_n \setminus \{e\} : j \in [n]\}$, so $\phi_1 = e$. Therefore we have $\mathbf{m}_\sigma^r \cdot \phi_1(j_2) = \mathbf{m}_\sigma^r(j_2) \neq \mathbf{m}_\sigma^r(i_2) = \mathbf{m}_\sigma^r \cdot \phi_2(j_2)$.

Finally, for Case IIC, without loss of generality we may assume that $i_1 < i_2$ and then split into the following four subcases:

- i) $(j_1 < i_2)$ and $(j_2 \geq i_2)$
- ii) $(j_1 < i_2)$ and $(j_2 < i_2)$
- iii) $(j_1 \geq i_2)$ and $(j_2 \geq i_2)$
- iv) $(j_1 \geq i_2)$ and $(j_2 < i_2)$.

However, since $\phi(i_2, j_2) \in T_n \setminus D^*(\mathbf{m}_\sigma^r)$ implies $i_2 \neq j_2$, subcases i) and iii) can be reduced to $(j_1 < i_2)$ and $(j_2 > i_2)$ and $(j_1 \geq i_2)$ and $(j_2 > i_2)$ respectively. For subcase i), we have $\mathbf{m}_\sigma^r \cdot \phi_1(j_1) = \mathbf{m}_\sigma^r(i_1) \neq \mathbf{m}_\sigma^r(j_1) = \mathbf{m}_\sigma^r \cdot \phi_2(j_1)$. Subcases ii) and iii) are solved in a similar manner. For subcase iv), if $j_1 > i_2$, then $\mathbf{m}_\sigma^r \cdot \phi_1(j_1) = \mathbf{m}_\sigma^r(i_1) \neq \mathbf{m}_\sigma^r(j_1) = \mathbf{m}_\sigma^r \cdot \phi_2(j_1)$. Otherwise, if $j_1 = i_2$, then $\phi_1 = \phi(i_1, i_2)$ and $\phi_1 = \phi(i_2, j_2)$. Thus if $\mathbf{m}_\sigma^r \cdot \phi_1 = \mathbf{m}_\sigma^r \cdot \phi_2$ then $\phi_1 \in D_{i_1}^*(\mathbf{m}_\sigma^r)$, which implies that $\phi_1 \notin T_n \setminus D^*(\mathbf{m}_\sigma^r)$, a contradiction. ■

Lemma 12 implies that we can calculate r -regular Ulam sphere sizes of radius 1 whenever we can calculate the appropriate duplication set. This calculation can be simplified by noting that for a sequence $\mathbf{m} \in \mathbb{Z}^n$ that $D(\mathbf{m}) \cap E(\mathbf{m}) = \emptyset$ (by the definition of $E(\mathbf{m})$) and then decomposing the duplication set into these components. This idea is stated in the next theorem

Theorem 13. $|S(\mathbf{m}_\sigma^r, 1)| = 1 + (n-1)^2 - |D(\mathbf{m}_\sigma^r)| - |E(\mathbf{m}_\sigma^r)|$.

Proof: By the definition of $D^*(\mathbf{m}_\sigma^r)$ and lemma 8,

$$\begin{aligned} & \{\mathbf{m}_\sigma^r \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) : \phi \in T_n \setminus D^*(\mathbf{m}_\sigma^r)\} \\ &= \{\mathbf{m}_\sigma^r \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) : \phi \in T_n \setminus D(\mathbf{m}_\sigma^r)\} \\ &= S(\mathbf{m}_\sigma^r, 1). \end{aligned}$$

This implies $|T_n \setminus D^*(\mathbf{m}_\sigma^r)| \geq |S(\mathbf{m}_\sigma^r, 1)|$. By lemma 12, for $\phi_1, \phi_2 \in T_n \setminus D^*(\mathbf{m}_\sigma^r)$, if $\phi_1 \neq \phi_2$, then $\mathbf{m}_\sigma^r \cdot \phi_1 \neq \mathbf{m}_\sigma^r \cdot \phi_2$. Hence we have $|T_n \setminus D^*(\mathbf{m}_\sigma^r)| \leq |S(\mathbf{m}_\sigma^r, 1)|$, which implies that $|T_n \setminus D^*(\mathbf{m}_\sigma^r)| = |S(\mathbf{m}_\sigma^r, 1)|$. It remains to show that $|T_n \setminus D^*(\mathbf{m}_\sigma^r)| = (n-1)^2 + 1 - |D(\mathbf{m}_\sigma^r)| - |E(\mathbf{m}_\sigma^r)|$. This is an immediate consequence of the fact that $|T_n| = (n-1)^2 + 1$ and $D(\mathbf{m}_\sigma^r) \cap E(\mathbf{m}_\sigma^r) = \emptyset$. ■

Theorem 13 reduces the calculation of $|S(\mathbf{m}_\sigma^r, 1)|$ to calculating $|D(\mathbf{m}_\sigma^r)|$ and $|E(\mathbf{m}_\sigma^r)|$. It is an easy matter to calculate $|D(\mathbf{m}_\sigma^r)|$, since it is exactly equal to $(n-2)$ times the number of $i \in [n]$ such that $\mathbf{m}_\sigma^r(i) = \mathbf{m}_\sigma^r(i-1)$ plus $(r-1)$ times the number of $i \in [n]$ such that $\mathbf{m}_\sigma^r(i) \neq \mathbf{m}_\sigma^r(i-1)$. We also showed how to calculate $|E(\mathbf{m})|$ earlier. The next example is an application of Theorem 13

Example. Suppose $\sigma := [1, 2, 3, 4, 9, 6, 7, 11, 5, 10, 12, 8]$. Then $\mathbf{m}_\sigma^3 := (1, 1, 1, 2, 3, 2, 3, 2, 4, 4, 3, 4)$. There are 3 values of $i \in [12]$ such that $\mathbf{m}_\sigma^3(i) = \mathbf{m}_\sigma^3(i-1)$, which implies that $|D(\mathbf{m}_\sigma^3)| = (3)(12-2) + (12-3)(3-1) = 48$. Meanwhile, by Lemmas 10 and 11, $|E(\mathbf{m}_\sigma^3)| = ((5-3)/2)((5-1)/2) = 2$. By Theorem 13, $|S(\mathbf{m}_\sigma^3, 1)| = (12-1)^2 - 48 - 2 = 71$.

VI. MIN/MAX SPHERES AND CODE SIZE BOUNDS

In this section we show choices of center achieving minimum and maximum r -regular Ulam sphere sizes for the radius $t = 1$ case. The minimum and maximum values are explicitly given. We then discuss resulting bounds on code size. First let us consider the r -regular Ulam sphere of minimal size.

Lemma 14. $|S(\mathbf{m}_e^r, 1)| \leq |S(\mathbf{m}_\sigma^r, 1)|$

Proof: In the case that $n/r = 1$, then $\mathbf{m}_e^r = e$ and $\mathbf{m}_\sigma^r = \sigma$, so that $|S(\mathbf{m}_e^r, 1)| = |S(\mathbf{m}_\sigma^r, 1)|$. Therefore we may assume that $n/r \geq 2$. By Theorem 13, $\min_{\sigma \in \mathbb{S}_n} (|S(\mathbf{m}_\sigma^r, 1)|) = 1 + (n-1)^2 - \max_{\sigma \in \mathbb{S}_n} (|D(\mathbf{m}_\sigma^r)| + |E(\mathbf{m}_\sigma^r)|)$. Since $n/r \geq 2$, we know that $n-2 > r-1$, which implies that for all $\sigma \in \mathbb{S}_n$, that $|D(\mathbf{m}_\sigma^r)|$ is maximized by maximizing the number of integers $i \in [n]$ such that $\mathbf{m}_\sigma^r(i) = \mathbf{m}_\sigma^r(i-1)$. This is accomplished by choosing $\sigma = e$, and hence for all $\sigma \in \mathbb{S}_n$, we have $|D(\mathbf{m}_e^r)| \geq |D(\mathbf{m}_\sigma^r)|$.

We next will show that for any increase in the size of $|E(\mathbf{m}_\sigma^r)|$ compared to $|E(\mathbf{m}_e^r)|$, that $|D(\mathbf{m}_\sigma^r)|$ is decreased

by a larger value compared to $|D(\mathbf{m}_e^r)|$, so that $(|D(\mathbf{m}_\sigma^r)| + |E(\mathbf{m}_\sigma^r)|)$ is maximized when $\sigma = e$.

Suppose $\sigma \in \mathbb{S}_n$. By Lemmas 10 and 11, $|E(\mathbf{m}_\sigma^r)|$ is characterized by the lengths of its locally maximal alternating substrings. For every locally maximal alternating substring $\mathbf{m}_\sigma^r[a, a+k-1]$ of \mathbf{m}_σ^r of length k , there are at least $k-2$ fewer instances where $\mathbf{m}_\sigma^r = \mathbf{m}_\sigma^r(i-1)$ when compared to instances where $\mathbf{m}_e^r(i) = \mathbf{m}_e^r(i-1)$. This is because for all $i \in [a+1, a+k-1]$, $\mathbf{m}_\sigma^r(i) \neq \mathbf{m}_\sigma^r(i-1)$. Hence for each locally maximal alternating substring $\mathbf{m}_\sigma^r(a, a+k-1)$, then $|D(\mathbf{m}_\sigma^r)|$ is decreased by at least $(k-2)(n-2-(r-1)) \geq (k-2)(r-1)$ when compared to $|D(\mathbf{m}_e^r)|$. Meanwhile, $|E(\mathbf{m}_\sigma^r)|$ is increased by the same locally maximal alternating substring by at most $(k-2)((k-2)/4)$ by Lemma 11. However, since $k \leq 2r$, we have $(k-2)((k-2)/4) \leq (k-2)(r-1)/2$, which is of course less than $(k-2)(r-1)$. ■

Lemma 14, along with Proposition 6 implies that the r -regular Ulam sphere size of radius $t=1$ is bounded (tightly) below by $(1 + (n-1)(n/r-1))$. This in turn implies the following sphere-packing type upper bound on any single error-correcting code.

Lemma 15. *Let C be a single-error correcting $\text{MPC}_o(n, r)$ code. Then*

$$|C|_r \leq \frac{n!}{(r!)^{n/r} (1 + (n-1)(n/r-1))}.$$

Proof: Let C be a single-error correcting $\text{MPC}_o(n, r)$ code. A standard sphere-packing bound argument implies that $|C|_r \leq \frac{n!}{(r!)^{n/r} (\min_{\sigma \in \mathbb{S}_n} (|S(\mathbf{m}_\sigma^r, 1)|))}$. The remainder of the proof follows from Proposition 6 and Lemma 14. ■

We have seen that $|S(\mathbf{m}_\sigma^r)|$ is minimized when $\sigma = e$. We now discuss the choice of center yielding the maximal sphere size. Let $\omega \in \mathbb{S}_n$ be defined as follows: $\omega(i) := ((i-1) \bmod (n/r))r + \lceil ir/n \rceil$ and $\omega := [\omega(1), \omega^*(2), \dots, \omega^*(n)]$. With this definition, for all $i \in [n]$, we have $\mathbf{m}_\omega^r(i) = i \bmod (n/r)$. For example, if $r=3$ and $n=12$, then $\omega = [1, 4, 7, 10, 2, 5, 8, 11, 3, 6, 9, 12]$ and $\mathbf{m}_\omega^r = (1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4)$. We can use Theorem 13 to calculate $|S(\mathbf{m}_\omega^r, 1)|$, and then show that this is the maximal r -regular Ulam sphere size (except for the case when $n/r=2$).

Lemma 16. *Let $n/r \neq 2$. Then $|S(\mathbf{m}_\sigma^r, 1)| \leq |S(\mathbf{m}_\omega^r, 1)|$ and if $n/r > 2$, then $|S(\mathbf{m}_\omega^r, 1)| = (1 + (n-1)^2) - (r-1)n$.*

Proof: Assume $n/r \neq 2$. First notice that if $n/r = 1$ then for any $\pi \in \mathbb{S}_n$ (including $\pi = \omega$), the sphere $S(\mathbf{m}_\pi^r, 1)$ contains exactly one element (the tuple of the form $(1, 1, \dots, 1)$). Hence the lemma holds trivially in this instance. Next, assume that $n/r > 2$. We will first prove that $|S(\mathbf{m}_\omega^r, 1)| = (1 + (n-1)^2) - (r-1)n$.

Since $n/r > 2$, it is clear that \mathbf{m}_ω^r contains no alternating subsequences of length greater than 2. Thus by Lemma 9, $E(\mathbf{m}_\omega^r) = \emptyset$ and therefore by Theorem 13, $|S(\mathbf{m}_\omega^r, 1)| = 1 + (n-1)^2 - |D(\mathbf{m}_\omega^r)|$. Since there does not exist $i \in [n]$ such that $\mathbf{m}_\omega^r(i) = \mathbf{m}_\omega^r(i-1)$, we have $|D(\mathbf{m}_\omega^r)| = (r-1)n$,

completing the proof of the first statement in the lemma.

We now prove that $|S(\mathbf{m}_\sigma^r, 1)| \leq |S(\mathbf{m}_\omega^r, 1)|$. Recall that $|D(\mathbf{m}_\sigma^r)|$ is equal to $(n-2)$ times the number of $i \in [n]$ such that $\mathbf{m}_\sigma^r(i) = \mathbf{m}_\sigma^r(i-1)$ plus $(r-1)$ times the number of $i \in [n]$ such that $\mathbf{m}_\sigma^r(i) \neq \mathbf{m}_\sigma^r(i-1)$. But $n/r > 2$ implies that $r-1 < n-2$, which implies $\min_{\pi \in \mathbb{S}_n} |D(\mathbf{m}_\pi^r, 1)| = (r-1)n$. Therefore

$$\begin{aligned} |S(\mathbf{m}_\sigma^r, 1)| &\leq 1 + (n-1)^2 - \min_{\pi \in \mathbb{S}_n} |D(\mathbf{m}_\pi^r, 1)| \\ &\quad - \min_{\pi \in \mathbb{S}_n} |E(\mathbf{m}_\pi^r, 1)| \\ &\leq 1 + (n-1)^2 - \min_{\pi \in \mathbb{S}_n} |D(\mathbf{m}_\pi^r, 1)| \\ &= 1 + (n-1)^2 - (r-1)n \\ &= |S(\mathbf{m}_\omega^r, 1)|. \end{aligned}$$

Extending the concept of perfect permutation codes discussed in [10], we define a perfect multipermutation code. Let C be an $\text{MPC}(n, r)$ code. Then C is a perfect t -error correcting code if and only if for all $\sigma \in \mathbb{S}_n$, there exists a unique $\mathbf{m}_c^r \in \mathcal{M}_r(C)$ such that $\mathbf{m}_\sigma^r \in S(\mathbf{m}_c^r, t)$. We call such C a **perfect t -error correcting $\text{MPC}(n, r)$** . With this definition the upper bound of lemma 16 implies a lower bound on a perfect single-error correcting $\text{MPC}(n, r)$.

Lemma 17. *Let $n/r \neq 2$ and let C be a perfect single-error correcting $\text{MPC}(n, r)$. Then*

$$|C|_r \geq \frac{n!}{(r!)^{n/r} ((1 + (n-1)^2) - (r-1)n)}.$$

Proof: Suppose $n/r \neq 2$ and C is a perfect single-error correcting $\text{MPC}(n, r)$. Then $\sum_{c \in C} |S(\mathbf{m}_c^r, 1)| = \frac{n!}{(r!)^{n/r}}$. This means

$$(|C|_r) \cdot \left(\max_{c \in C} (|S(\mathbf{m}_c^r, 1)|) \right) \geq \frac{n!}{(r!)^{n/r}},$$

which by Lemma 16 implies the desired result. ■

A more general lower bound is easily obtained by applying Lemma 16 with a standard Gilbert-Varshamov bound argument. In the lemma statement, C is an $\text{MPC}_o(n, r, d)$ if and only if C is an $\text{MPC}(n, r)$ such that $\min_{\sigma, \pi \in C, \sigma \neq \pi} d_o^r(\sigma, \pi) = d$.

Lemma 18. *Let $n/r \neq 2$ and C be an $\text{MPC}_o(n, r, d)$ code of maximal cardinality. Then*

$$|C|_r \geq \frac{n!}{(r!)^{n/r} (1 + (n-1)^2 - (r-1)n)^{d-1}}$$

Proof: Suppose that $n/r \neq 2$ and that C is an $\text{MPC}_o(n, r, d)$ code of maximal cardinality. For all $\sigma \in \mathbb{S}_n$, there exists $c \in C$ such that $d_o^r(\sigma, c) \leq d-1$. Otherwise, we could add $\sigma \notin C$ (and its entire equivalence class $R_r(\sigma)$) to C while maintaining a minimum distance of d , contradicting the assumption that $|C|_r$ is maximal.

Therefore $\bigcup_{c \in C} S(\mathbf{m}_c^r, d-1) = \mathcal{M}_r(\mathbb{S}_n)$. This in turn implies

that

$$\sum_{c \in C} |S(\mathbf{m}_c^r, d-1)| \geq \frac{n!}{(r!)^{n/r}}.$$

Of course, the left hand side of the above inequality is less than or equal to $(|C|_r) \cdot \left(\max_{c \in C} (|S(\mathbf{m}_c^r, d-1)|) \right)$. Hence Lemma 16 implies that

$$(1 + (n-1)^2 - (r-1)n)^{d-1} \geq \max_{c \in C} (|S(\mathbf{m}_c^r, d-1)|),$$

so the conclusion holds. \blacksquare

VII. CONCLUSION

This paper compared the Ulam metric for the permutation and multipermutation cases, providing a simplification of the r -regular Ulam metric. The surprising fact that r -regular Ulam sphere sizes differ depending upon the center was also shown. New methods for calculating the size of r -regular Ulam sphere sizes were provided, first using Young Tableaux for spheres of any radius centered at \mathbf{m}_e^r . Another method used duplicate translocation sets to calculate sphere sizes for a radius of $t = 1$ for any center. Resulting bounds on Code size were also provided. Many open questions remain, including the existence of perfect codes, sphere size calculation methods for more general parameters, and tighter bounds on code size.

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REFERENCES

- [1] A. Barg and A. Mazumdar, "Codes in Permutations and Error Correction for Rank Modulation," in Proc. IEEE ISIT, pp 854-858, 2010
- [2] I.F. Blake, G. Cohen, and M. Deza, "Coding with permutations," in Inform. Contr., vol. 43, pp. 1-19, 1979.
- [3] F. Farnoud, V. Skachek, and O. Milenkovic, "Error-correction in flash memories via codes in the Ulam metric," in IEEE Trans. Inf. Theory, Vol. 59 No. 5, pp. 3003-3030, 2013.
- [4] F. Farnoud and O. Milenkovic, "Multipermutation Codes in the Ulam Metric," in Proc. IEEE ISIT, pp. 2754-2758, 2014.
- [5] J. Frame, G. Robinson, and R. Thrall, "The hook graphs of the symmetric group," in Canad. J. Math., 6, pp. 316-324, 1954.
- [6] W. Fulton, "Young Tableaux," London Mathematical Society Student Texts. Cambridge University Press, 1997.
- [7] F. Gologlu, J. Lember, A. Riet, V. Skachek, "New Bounds for Permutation Codes in Ulam Metric," in Proc. IEEE ISIT, pp. 1726-1730, 2015.
- [8] A. Jiang, R. Mateescu, M. Schwartz, J. Bruck, "Rank Modulation for Flash Memories," in Proc. IEEE ISIT, pp. 1731-1735, 2008.
- [9] A. Jiang, M. Schwartz, and J. Bruck, "Correcting Charge-Constrained Errors in the Rank-Modulation Scheme," in IEEE Trans. Inf. Theory, Vol. 56 No. 5, pp. 2112-2120, 2010.
- [10] J. Kong, M. Hagiwara, "Nonexistence of Perfect Permutation Codes in the Ulam Metric," Proc. of ISITA 2016, Monterey, USA, 2016.
- [11] F. Lim and M. Hagiwara, "Linear programming upper bounds on permutation code sizes from coherent configurations related to the Kendall-tau distance metric," in Proc. IEEE ISIT, pp. 2998-3002, 2012.
- [12] C. Schensted, "Longest increasing and decreasing subsequences," Canad. J. Math. 13, pp. 179-191, 1961.
- [13] D. Slepian, "Permutation modulation," Proc. IEEE, 53, 228-237 (1965)
- [14] R. P. Stanley, "Algebraic Combinatorics. Walks, Trees, Tableaux, and More," Springer Science+Business Media New York, 2013.